On equations of double planes with $p_g = q = 1$

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Abstract

This paper describes how to compute equations of plane models of minimal Du Val double planes of general type with $p_g = q = 1$ and $K^2 = 2, \ldots, 8$. A double plane with $K^2 = 8$ having bicanonical map not composed with the associated involution is also constructed. The computations are done using the algebra system Magma.
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1 Introduction

There are two typical methods to construct examples of surfaces: Campedelli — double covers ramified over curves, usually highly singular — and Godeaux — quotients by group actions (*cf.* [Re]). In this paper the first method is used to obtain new examples of double planes of general type with $p_g = q = 1$. Often the singularities impose too many conditions on the linear parameters of systems of curves, and this implies hard calculations. The *Computational Algebra System MAGMA* ([BCP]) is used to perform the computations.

Several authors have studied surfaces of general type with $p_g = q = 1$ ([Ca1], [Ca2], [CC1], [CC2], [CP], [Po1], [Po2], [Po3], [Po4], [Pi]), but these surfaces are still not completely understood, and few examples are known.

For such a smooth minimal surface S, one has $2 \leq K^2 \leq 9$ and $K^2 = 9$ only if the bicanonical map of S (given by $|2K|$) is birational (from [CM] and [Xi1, Théorème 2.2]). Since $q = 1$, the Albanese map of S is a connected fibration onto an elliptic curve. We denote by g the genus of a general Albanese fibre of S.

Du Val ([Du]) classified the regular surfaces S of general type with $p_g \geq 3$ whose general canonical curve is smooth and hyperelliptic. For these surfaces the bicanonical map is composed with an involution i such that S/i is rational. The families of surfaces exhibited by Du Val are nowadays called the *Du Val examples*.

The *standard case* of non-birationality of the bicanonical morphism ϕ_2 is the case where S has a genus 2 fibration.

For the non-standard case with ϕ_2 of degree 2 and bicanonical image a ruled surface, Xiao Gang ([Xi2, Theorem 2]) extended Du Val's list to $p_g(S) \geq 1$ and added two extra families (this result is still valid assuming only that ϕ_2 is composed with an involution such that the quotient surface is a ruled surface). Later G. Borrelli ([Br]) excluded these two families, confirming that the only possibilities for this case are the Du Val examples.

This way, if the bicanonical map ϕ_2 of S is composed with i, S/i is ruled and S presents the non-standard case, then S is a Du Val double plane (see

Definition 3). But, to my knowledge, the existence of these double planes with $p_g = q = 1$ and $K^2 = 4, \ldots, 7$ has not been shown yet, and no equation has been given for $K^2 = 3$ or 8.

In this paper we compute equations of plane models of minimal Du Val double planes of general type with $p_g = q = 1$ and $K^2 = 2, \ldots, 8$. We also construct a double plane with $K^2 = 8$ having bicanonical map not composed with the associated involution.

Notice that we give the first example of a minimal surface of general type with $p_g = q = 1$ and $K^2 = 7$. We also give the first example with $K^2 = 5$ having Albanese fibration of genus $g \neq 2$.

The first example with $K^2 = 6$ has been given by the author in [Ri], as a double cover of a $K3$ surface. Polizzi ([Po2]) also gives examples with $K^2 = 6, 4$ and $g \neq 2$. These surfaces contain $8 - K^2$ (-2)-curves, all in the same Albanese fibre. One can verify that, in the double plane examples with $K^2 = 4$ or 6, there is no Albanese fibre containing $8 - K^2$ (-2)-curves. Therefore our surfaces are different from the above examples.

In [Po4], Polizzi classifies surfaces of general type with $p_g = q = 1$, $K^2 = 8$ and bicanonical map of degree 2. He constructs examples using quotients under the action of a group and shows that these surfaces are Du Val double planes, describing the branch locus of the respective plane models. In sections 3.1 and 3.4 we show how to obtain equations for such branch loci.

Section 2 contains a structure description of the Albanese fibration of S and the definition of Du Val double plane. In Section 3 we describe the principal steps of the constructions. To obtain the equations of the branch loci we impose conditions to the linear parameters of systems of plane curves. Since the ramification curves are contained in Albanese fibres, in some cases it is easier to start by constructing non-reduced Albanese fibres, which simplify the computations. These are done using the algebra system Magma in Appendix A.

Notation and conventions

We work over the complex numbers; all varieties are assumed to be projective algebraic. For a projective smooth surface S, the *canonical class* is denoted by K, the *geometric genus* by $p_g := h^0(S, \mathcal{O}_S(K))$, the *irregularity* by $q := h^1(S, \mathcal{O}_S(K))$ and the *Euler characteristic* by $\chi = \chi(\mathcal{O}_S) = 1 + p_g - q$.

An $(-n)$-*curve* C on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $C^2 = -n$. We say that a curve singularity is *negligible* if it is either a double point or a triple point which resolves to at most a double point after one blow-up. An $(m_1, m_2, \ldots)$-point, or point of order $(m_1, m_2, \ldots)$, is a point of multiplicity m_1, which resolves to a point of multiplicity m_2 after one blow-up, etc.

An *involution* of a surface S is an automorphism of S of order 2. We say that a map is *composed with an involution i* of S if it factors through the double cover $S \to S/i$.

The rest of the notation is standard in algebraic geometry.

Acknowledgements

2 Preliminaries

We say that a smooth surface S is a *double plane* if S has an involution i such that S/i is a rational surface. A *plane model* of S is a double cover $X \to \mathbb{P}^2$ such that X is a normal surface and there exists a commutative diagram

$$
\begin{array}{ccc}
S & \longrightarrow & X \\
\downarrow & & \downarrow \\
S/i & \longrightarrow & \mathbb{P}^2
\end{array}
$$

such that the horizontal arrows denote birational maps.

2.1 Double planes with $q = 1$

Let S be a smooth minimal surface of general type with an involution i. Since S is minimal of general type, this involution is biregular. The fixed locus of i is the union of a smooth curve R'' (possibly empty) and of $t \geq 0$ isolated points $P_1, \ldots, P_t$. Let S/i be the quotient of S by i and $p : S \to S/i$ be the projection onto the quotient. The surface S/i has nodes at the points $Q_i := p(P_i)$, $i = 1, \ldots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via p of R'' is a smooth curve B'' not containing the singular points Q_i, $i = 1, \ldots, t$. Let now $h : V \to S$ be the blow-up of S at $P_1, \ldots, P_t$ and set $R' = h^*(R'')$. The involution i induces a biregular involution $\widetilde{i}$ on V, whose fixed locus is $R := R' + \sum_1^t h^{-1}(P_i)$, and the quotient $W := V/\widetilde{i}$ is smooth.

Suppose now that $q(S) = 1$ and S/i is a rational surface. Since $q = 1$, the Albanese variety of S is an elliptic curve E and the Albanese map is a connected fibration (see *e.g.* [Be] or [BPV]). This fibration is preserved by i and so we have a commutative diagram

$$
\begin{array}{ccccc}
V & \xrightarrow{\ h\ } & S & \longrightarrow & E \\
{\scriptstyle \pi}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow \\
W & \longrightarrow & S/i & \longrightarrow & \mathbb{P}^1
\end{array}
\qquad (1)
$$

Denote by

$$
f_A : W \to \mathbb{P}^1
$$

the fibration induced by the Albanese fibration of S. The double cover $E \to \mathbb{P}^1$ is ramified over 4 points p_j of $\mathbb{P}^1$, thus the branch locus B of $\pi : V \to W$ is contained in 4 fibres

$$
F_A^j := f_A^*(p_j), \ j = 1, ..., 4,
$$

of the fibration f_A. By Zariski's Lemma (see *e.g.* [BPV]) the irreducible components B_i of B satisfy $B_i^2 \leq 0$. Since $\pi^*(F_A^j)$ has even multiplicity, each component of F_A^j which is not a component of the branch locus must be of even multiplicity.

In the following sections W can always be contracted to $\mathbb{P}^2$. We keep the same notation for the image of f_A and F_A^j on $\mathbb{P}^2$.

2.2 Du Val double planes

Notation 1 *Given distinct points $p_0, \ldots, p_j, \ldots, p_{j+s} \in \mathbb{P}^2$, let T_i denote the line through p_0 and p_i, $i = 1, \ldots, j$. We say that a plane curve is of type*

$$d\big(m, (n,n)_T^j, r^s\big)$$

if it is of degree d and if it has: an m-uple point at p_0, an (n,n)-point at $p_1, \ldots, p_j$, an r-uple point at $p_{j+1}, \ldots, p_{j+s}$ and no other non-negligible singularities. The index $_T$ is used if T_i is tangent to the (n,n)-point at p_i.

An obvious generalization is used if there are other singularities.

Definition 2 *Let C_0 and F denote, respectively, the negative section and a rational fibre of the Hirzebruch surface $\mathbb{F}_2$.*

The Bombieri-Du Val surface is the minimal model of the double cover of $\mathbb{F}_2$ with branch locus a smooth curve in $C_0 + |7C_0 + 14F|$.

Definition 3 *A Du Val surface is either*

$\mathcal{B}$*) the Bombieri-Du Val surface*

or a minimal double plane having a plane model with branch locus D one of the following:

$\mathcal{D}$*) a smooth curve of degree 8;*

$\mathcal{D}_0$*) a smooth curve of degree 10;*

$\mathcal{D}_n$*) a curve of type $[10 + 2n]\big(2n + 2, (5,5)_T^n\big)$, $n \in \{1, \ldots, 6\}$;*

or one of $\mathcal{B}$, $\mathcal{D}$ or $\mathcal{D}_n$, imposing additional 4-uple or $(3,3)$-points to D.

Surfaces of type $\mathcal{B}$, $\mathcal{D}$ or $\mathcal{D}_n$ are called *Du Val's ancestors*. The Bombieri surface has $K^2 = 9$, $p_g = 6$ and $q = 0$; a Du Val ancestor of type $\mathcal{D}$ has $K^2 = 2$, $p_g = 3$ and $q = 0$; those of type $\mathcal{D}_n$, $n \geq 0$, have $K^2 = 8$, $p_g = 6 - n$ and $q = 0$, except possibly in the case $n = 6$, where $p_g = q = 1$ if the 6 singular points are contained in a conic.

Notice that the imposition of a 4-uple point to the branch locus decreases K^2 by 2 and the Euler characteristic χ by 1, while a $(3,3)$-point decreases both K^2 and χ by 1. Negligible singularities in the branch locus do not change these invariants.

The imposition of $6 - n$ 4-uple or $(3,3)$-points to the branch locus of a Du Val surface of type $\mathcal{D}_n$ gives $p_g = 0$ if the singular points $p_1, \ldots, p_6$ are not contained in a conic and $p_g = q = 1$ otherwise.

Notice that, in the case $\mathcal{D}_n$, the branch locus contains the lines $T_1, \ldots, T_n$. Each of these lines corresponds to two (-2)-curves in W, which contract to two nodes of S/i. A $(3,3)$-point in the branch locus also corresponds to a node of S/i.

The bicanonical map of a Du Val surface factors through a map of degree 2 onto a rational surface. For more information on Du Val surfaces see [Du] or [Ci].

2.3 The Magma procedures *LinSys* and *LinSys2*

Magma has the function *LinearSystem* which calculates linear systems of plane curves with ordinary singularities, but we want to work with non-ordinary singularities. To achieve this, we define two procedures. The first one, *LinSys*, calculates the linear system L of plane curves of degree d, in an affine plane $\mathbb{A}$, having singular points p_i of order $(m1_i, m2_i)$ with tangent slope td_i.

The other procedure, *LinSys2*, calculates the sub-system J, of a given linear system L of plane curves, of those sections which have a singularity, at a point q, of type $m = (m_1, \ldots, m_j)$ with tangent slopes given by $td = [td_1, \ldots, td_{j-1}]$.

The code lines for these procedures are in Appendix A.1.

3 Equations of plane models with $p_g = q = 1$

In this section we obtain equations of the branch locus $B \subset \mathbb{P}^2 = \mathbb{P}(1,1,1)$ of minimal Du Val double planes S of type $\mathcal{D}_n$ (see Definition 3) with $p_g = q = 1$ and $K^2 = 2, \ldots, 8$. Notice that if $f = 0$ is the defining equation of B, then the equation $w^2 = f$ gives a plane model of S, in the weighted projective space $\mathbb{P}((\deg f)/2, 1, 1, 1)$.

In Section 3.5 we construct a double plane with $K^2 = 8$ having bicanonical map not composed with the associated involution.

Table (2) lists the type of each branch curve that we are going to obtain and the corresponding values of (K^2, g), where g denotes the genus of a general Albanese fibre of S.

We keep Notation 1.

Type of branch curve	(K^2, g)
$22\big(14, (5,5)_T^6\big)$	$(8,5),\ (8,4),\ (8,3)$
$[10+2n]\big(2n+2, (5,5)_T^n, 4^{6-n}\big)$ $n = 5, 4, 3$	$(6,4),\ (4,3),\ (2,2)$
$[10+2n]\big(2n+2, (5,5)_T^n, (3,3), 4^{5-n}\big)$ $n = 5, 4, 3$	$(7,5),\ (5,4),\ (3,3)$
$18\big(10, (5,5)_T^4, (3,3)^2\big)$	$(6,3)$
$26\big(14, (7,7)_T^5\big)$	$(8,4)$

$$(2)$$

First we find double planes with $(K^2, g) = (8,5)$ and $(K^2, g) = (7,5)$ such that the branch locus contains an element of the pencil f_A which induces the Albanese fibration. From here constructions with $(K^2, g) = (6,4), (5,4), (4,3), (3,3), (2,2)$ will follow easily. A construction with $(K^2, g) = (6,3)$ and branch locus strictly contained in elements of f_A is also given. Then we get surfaces with $(K^2, g) = (8,4)$ or $(8,3)$. These three surfaces with $K^2 = 8$ are surfaces of type I, II, III described by F. Polizzi in [Po4]. Finally we construct a pencil l of plane curves

such that the minimal double plane with branch locus a general element of l has $K^2 = 8$ and bicanonical map not composed with the corresponding involution.

In order to obtain $q = 1$, all the singularities, except the one at p_0, are chosen to be contained in a conic.

A difficulty that arises in the computation of singular curves is that often the computer is not able to finish the calculations. Except for Section 3.2, our method is to try to figure out the configuration of the non-reduced fibres F_A^i of f_A, described in Section 2.1. The problem of finding the support of F_A^i deals with curves of lower degree and with simpler singularities, hence with faster computations.

The next sections of this chapter describe the principal steps. The detailed calculations are done in Appendix A, using the Computational Algebra System Magma.

We keep Notation 1.

3.1 $\quad K^2 = 8, 6, 4, 2$ **and** $g = 5, 4, 3, 2$

To construct a Du Val's ancestor S of type $\mathcal{D}_6$ (it has $K^2 = 8$) it suffices to find a curve B' of type

$$16\left(8, (4,4)_T^6\right),$$

singular at $p_0, \ldots, p_6$, such that the branch locus $B := B' + \sum_1^6 T_i$ is reduced. If $p_1, \ldots, p_6$ are contained in a conic, then $p_g(S) = q(S) = 1$.

In this example the pencil f_A which induces the Albanese fibration of S has the following elements:

$$F_A^1 = 2D_1 + T_3 + \cdots + T_6, \quad F_A^2 = 2D_2 + T_1 + T_2, \quad F_A^3 = 2D_3, \quad F_A^4 = D_4,$$

where $D_1, \ldots, D_4$ are curves of types

$$6\left(2, (2,2)_T^2, (2,1)_T^4\right), \quad 7\left(3, (2,1)_T^2, (2,2)_T^4\right), \quad 8\left(4, (2,2)_T^6\right), \quad 16\left(8, (4,4)_T^6\right),$$

respectively.

To construct B', we first find points $p_0, \ldots, p_6$ such that D_1 exists. Then we use the procedure $LinSys$ (see Section 2.3) to verify the existence of D_2 and D_3, also singular at $p_0, \ldots, p_6$. The curve $B' = D_4$ is a general element of the pencil generated by D_1, D_2, D_3.

So let us look for D_1. Briefly, the steps are as follows. Let $\mathbb{A}$ be an affine plane, C be a smooth conic not containing the origin p_0 of $\mathbb{A}$ and $p_1, \ldots, p_4$ be points in C. Denote by L the linear system of plane curves of type

$$6\left(2, (2,2)_T^2, (2,1)_T^2\right),$$

with singularities at $p_0, \ldots, p_4$, respectively. Let F be a general element of L and p_5, p_6 be general points of $\mathbb{A}$. We define a scheme Sch by imposing the following conditions:

· $p_5, p_6 \in C \bigcap F$;

· p_5, p_6 are double points of F;

· the singularities of F at p_5, p_6 have one branch tangent to T_5, T_6;

· $p_5 \neq p_6$ and $p_5, p_6 \notin \{p_0, \ldots, p_4\}$.

Now we compute the points of Sch with Magma, choosing one of the solutions for p_5, p_6, and we use the procedure $LinSys$ to compute B', a general element of the pencil of curves of type $16(8, (4, 4)_T^6)$, with singularities at $p_0, \ldots, p_6$.

Finally we perform some verifications: that B' is reduced, the singularities are as expected, etc.

With this we find a minimal double plane with $p_g = q = 1$ and $K^2 = 8$. The divisor $2D_1 + T_3 + \cdots + T_i$ is also a good candidate for one of the singular fibres F_A^i in the case where the branch locus is a curve of type

$$[10 + 2n]\left(2n + 2, (5, 5)_T^n, 4^{6-n}\right), \quad n = 5, 4, 3.$$

In fact, using the points $p_0, \ldots, p_6$ and the procedure $LinSys$, one can find branch loci of those types, obtaining then minimal double planes with $p_g = q = 1$, $K^2 = 6, 4, 2$ and $g = 4, 3, 2$, respectively.

The corresponding Magma calculations are in Appendix A.2.1. There we use symmetry in order to obtain faster computations.

3.2 $K^2 = 7, 5, 3$ and $g = 5, 4, 3$

Here we impose a $(3, 3)$-point to the branch locus of a Du Val's ancestor of type $\mathcal{D}_5$, *i.e.* we construct a plane curve B' of type

$$15(7, (4, 4)_T^5, (3, 3))$$

such that $B := B' + \sum_1^5 T_i$ is reduced (see Notation 1). The double cover with branch locus B is a plane model of a Du Val double plane S with $K^2 = 7$ and $\chi = 1$.

In this example the $(3, 3)$-point p_6 is infinitely near to the $(5, 5)$-point p_1 of B (*i.e.* there is a $(5, 5, 3, 3)$-point at p_1) and the line T_1, through p_0, p_1, is tangent to the conic C defined by $p_1, \ldots, p_5$. This implies $q(S) = 1$, because $\widetilde{C} - \sum_1^6 E_i$ is effective, where the curves E_i are the exceptional divisors corresponding to the blow-ups at the points p_i and $\widetilde{C}$ is the pullback of C.

In this case the pencil f_A has elements

$$F_A^1 = 2D_1 + T_4 + T_5, \quad F_A^2 = 2D_2 + T_2 + T_3, \quad F_A^3 = 2D_3, \quad F_A^4 = D_4 + T_1,$$

where $D_1, \ldots, D_4$ are curves of types

$$7\left(3, (2, 2)_T^3, (2, 1)_T^2, (2, 2)\right), \quad 7\left(3, (2, 2)_T, (2, 1)_T^2, (2, 2)_T^2, (2, 2)\right),$$

$$8\left(4, (2, 2)_T^5, (2, 2)\right), \quad 15(7, (4, 4)_T^5, (3, 3)),$$

respectively.

To find $B' = D_4$ we proceed as follows. In an affine plane $\mathbb{A}$, we fix a smooth conic C not containing the origin p_0 and we choose distinct points $p_1, \ldots, p_5 \in C$ such that T_1 is tangent to C at p_1. We compute the linear system L of plane curves of type $15(7, (4, 4)_T^5)$ and we resolve the base point of L at p_1, denoting the resulting linear system by L_1. The defining polynomial of B' is given by (the blow-down of) a linear combination of elements of L_1.

In order to obtain a $(3,3)$-point p_6 (infinitely near to p_1), we impose conditions, given by the zeros of the derivatives up to order 2, to the elements of L_1. Then we resolve this point and impose another triple (infinitely near) point. With all these conditions we define a matrix, which is denoted by Mt in Appendix A.2.2. To have a solution it is necessary that Mt has no maximal rank.

We define a scheme Sch by imposing the following conditions:

· the vanishing of the maximal minors of Mt;

· the points are infinitely near.

Now we compute the points of Sch with Magma, choosing one of the solutions for p_6, and we use the procedures of Section 2.3 to compute B', with singularities at $p_0, \ldots, p_6$. Finally we perform some verifications: that B' is reduced, the singularities are as expected, etc.

Notice that there is no need to verify the non-existence of other singularities: in the presence of another non-negligible singularity, the computation of the invariants of the corresponding double plane (using the formulas of [BPV, Section V. 22]) leads to a contradiction.

To find the pencil which induces the Albanese fibration, one uses again the procedures $LinSys$ and $LinSys2$ to compute the pencil of curves of type

$$16(8, (4,4)_T^5, (4,4)),$$

through $p_0, \ldots, p_6$, and to verify that the elements $F_A^1, \ldots, F_A^4$ are as claimed.

With this we find a minimal double plane with $p_g = q = 1$, $K^2 = 7$ and $g = 5$. Using the above Magma procedures, one can verify that there exist also branch loci of type

$$18 \left(10, (5,5)_T^4, 4, (3,3) \right) \quad \text{and} \quad 16 \left(8, (5,5)_T^3, 4^2, (3,3) \right),$$

through $p_0, \ldots, p_6$. These correspond to minimal Du Val double planes with $p_g = q = 1$ and $(K^2, g) = (5, 4), (3, 3)$. The pencils which induce the Albanese fibration are of type

$$15 \left(7, (4,4)_T^4, 4, (4,4) \right) \quad \text{and} \quad 14 \left(6, (4,4)_T^3, 4^2, (4,4) \right),$$

respectively.

The Magma calculations for this section are in Appendix A.2.2.

3.3 $K^2 = 6$ and $g = 3$

In this section we impose two $(3,3)$-points to the branch locus of a Du Val's ancestor of type $\mathcal{D}_4$, *i.e.* we construct a plane curve B' of type

$$14(6, (4,4)_T^4, (3,3)^2),$$

singular at points $p_0, \ldots, p_6$, such that $B := B' + \sum_1^4 T_i$ is reduced. In this example the $(3,3)$-points p_5, p_6 are tangent to the conic C through $p_1, \ldots, p_6$.

The double cover with branch locus B is a plane model of a Du Val double plane S with $K^2 = 6$, $p_g = q = 1$ and $g = 3$.

The pencil f_A which induces the Albanese fibration of S contains four fibres

$$F_A^1 = D_1 + 2T_0, \quad F_A^2 = D_2, \quad F_A^3 = 2D_3, \quad F_A^4 = 2C + T_1 + \cdots + T_4,$$

where T_0 is a line through p_0, p_5, p_6 and D_1, D_2, D_3 are curves of types

$$6\left(2, (2, 2)_T^4, (1, 1)^2\right), \quad 8\left(4, (2, 2)_T^4, (2, 2)^2\right), \quad 4\left(2, (1, 1)_T^4, (1, 1)^2\right),$$

respectively. We have
$$B' = D_1 + D_2.$$

First we construct D_1 and then we use the procedure $LinSys$ of Section 2.3 to obtain D_2 (a general element of f_A) and to verify the existence of D_3.

To find D_1, we follow the steps used in Section 3.1, but now with L the linear system of curves of type $6\left(2, (2, 2)_T^4\right)$ and Sch defined by

· $p_5, p_6 \in C \bigcap F$;

· F is smooth at p_5, p_6;

· F is tangent to C at p_5, p_6;

· $p_5 \neq p_6$ and $p_5, p_6 \notin \{p_0, \ldots, p_4\}$.

The details can be found in Appendix A.2.3. Again we use symmetry in order to increase speed of calculations.

3.4 $K^2 = 8$ and $g = 4$ or 3

Given, in an affine plane $\mathbb{A}$, a general plane cubic Q not containing the origin p_0, there are 6 points $p_1, \ldots, p_6 \in Q$ such that each line T_i, defined by p_0, p_i, is tangent to Q. In this example we have

$$F_A^1 = 3D_1, \quad F_A^2 = D_2, \quad F_A^3 = 2D_3, \quad F_A^4 = 2Q + T_1 + \ldots + T_6,$$

where D_1, D_2, D_3 are curves of types

$$4\left(2, (1, 1)_T^6\right), \quad 12\left(6, (3, 3)_T^6\right), \quad 6\left(3, (2, 1)_T^6\right),$$

respectively.

Using the procedure $LinSys$ (see Section 2.3), one can verify the existence of the curves D_1, D_2, D_3, through $p_0, \ldots, p_6$, and that $p_1, \ldots, p_6$ are contained in a conic. The double cover with branch locus $D_1 + D_2 + \sum_1^6 T_i$ is a plane model of a Du Val double plane S with $K^2 = 6$, $p_g = q = 1$ and $g = 4$. This is a surface of type II described in [Po4].

Finally the surface of type I described in [Po4]:
Let $C \subset \mathbb{P}^2$ be a smooth conic and $p_0 \notin C$, $p_1, \ldots, p_6 \in C$ be points such that the lines T_5 and T_6, through p_0, p_5 and p_0, p_6, are tangent to C. One can verify the existance of a smooth cubic D of type $3(1, (1, 1)_T^4, 1^2)$, through $p_0, \ldots, p_6$. Let
$$F_A^1 = 2C + T_1 + \cdots + T_4, \qquad F_A^2 = 2D + T_5 + T_6$$

and F_A^3, F_A^4 be general elements of the pencil generated by F_A^1 and F_A^2. The branch locus in this case is

$$F_A^3 + F_A^4 + \sum_1^6 T_i.$$

3.5 $K^2 = 8$, $g = 4$, ϕ_2 **not composed with** i

This section describes the steps to obtain an equation of a double plane S of general type with $K^2 = 8$, $p_g = q = 1$ and Albanese fibres of genus $g = 4$, such that the bicanonical map ϕ_2 of S is not composed with the associated involution i of S.

In this example the pencil f_A which induces the Albanese fibration has four fibres

$$F_A^1 = 2D_1 + T_5, \quad F_A^2 = 6D_2 + 3T_1, \quad F_A^3 = 2D_3 + T_2 + T_3 + T_4, \quad F_A^4 = D_4,$$

where $D_1, \ldots, D_4$ are curves of types

$$10\left(4, (3,3)_T^4, (3,2)_T\right), \qquad 3\left(1, 1, (1,1)_T^4\right),$$

$$9\left(3, (3,3), (3,2)_T^3, (3,3)\right), \qquad 21\left(9, (6,6)_T^5\right),$$

respectively. The surface S is the minimal double plane given by the branch locus

$$D_4 + \sum_1^5 T_i.$$

From the double cover formulas ([BPV, Section V. 22]), one obtains $K_S^2 = 8$ (notice that the pullback of the lines T_i contains 10 (-1)-curves), $\chi(\mathcal{O}_S) = 1$ and $p_g(S)$ is equal to the number of generators of the linear system of curves of type $10(6, (2,3)_T^5)$.

The points $p_0, \ldots, p_5$ are chosen such that the line T_1, through p_0, p_1, is tangent to the conic defined by $p_1, \ldots, p_5$. One can verify that this implies the existence of the cubic D_2. After finding D_1, we verify the existence of D_2 and D_3. The curve D_4 is a general element of the pencil generated by D_1, D_2, D_3. One can use the procedure $LinSys$ to compute the number of elements of the linear system of curves of type $10(6, (2,3)_T^5)$, showing that $p_g(S) = 1$.

The difficulty here is the computation of D_1. This is done following the steps of Section 3.2. We omit the details.

A Appendix: Magma computations

In this appendix several computations are done using the *Computational Algebra System MAGMA*.

The following functions will be useful:

```
function D(F,i);P:=Parent(F);
return Derivative(F,P.i);end function;


function D2(F,i,j);P:=Parent(F);
return Derivative(Derivative(F,P.i),P.j);end function;
```

i.e. from now on D(F,i) means $\frac{\partial F}{\partial P.i}$ and D2(F,i,j) means $\frac{\partial^2 F}{\partial P.j \partial P.i}$.

A.1 The procedures *LinSys* and *LinSys2*

```
procedure LinSys(A,d,p,m1,m2,td,~L)
  //p,m1,... are tuples of p_i,m1_i,...
  x:=A.1;y:=A.2;//The coordinates of A.
  L:=LinearSystem(LinearSystem(A,d),p,m1);
  for j:=1 to #m2 do
    if #Sections(L) eq 0 then break;end if;
    a:=p[j][1];b:=p[j][2];
    Bup:=[Evaluate(Sections(L)[i],y,(x-a)*y+b) div (x-a)^m1[j]:\
    i in [1..#Sections(L)]];//The strict transform of the
    //blown-up curves.
    L1:=LinearSystem(A,Bup);
    L2:=LinearSystem(L1,A![a,td[j]],m2[j]);//Imposing the
    //infinitely near singularity.
    if #Sections(L2) eq 0 then L:=L2;break;end if;
    Bdn:=[Evaluate((x-a)^m1[j]*Sections(L2)[i],y,(y-b)/(x-a)):\
    i in [1..#Sections(L2)]];//The blown-down curves.
    R:=Universe(Bdn);
    //R is a Rational function field. We need an homomorphism
    //to send the elements of Bdn into a polynomial ring.
    h:=hom<R->CoordinateRing(A)|[x,y]>;
    L:=LinearSystem(A,[h(Bdn[i]):i in [1..#Bdn]]);
  end for;
end procedure;

procedure LinSys2(A,L,q,m,td,~J)
  x:=A.1;y:=A.2;//The coordinates of A.
  J:=LinearSystem(L,q,m[1]);
  td:=[q[2]] cat td;
  for j:=1 to #td-1 do
    if #Sections(J) eq 0 then break;end if;
    b:=td[j];
    Bup:=[Evaluate(Sections(J)[i],y,(x-q[1])*y+b) div\
    (x-q[1])^m[j]:i in [1..#Sections(J)]];
    J1:=LinearSystem(A,Bup);
    J:=LinearSystem(J1,A![q[1],td[j+1]],m[j+1]);
  end for;
  //
  for j:=#td-1 to 1 by -1 do
    if #Sections(J) eq 0 then break;end if;
    b:=td[j];
    Bdn:=[Evaluate((x-q[1])^m[j]*Sections(J)[i],y,(y-b)/\
    (x-q[1])):i in [1..#Sections(J)]];
    R:=Universe(Bdn);
    h:=hom<R->CoordinateRing(A)|[x,y]>;
    J:=LinearSystem(A,[h(Bdn[i]):i in [1..#Bdn]]);
  end for;
```

```
end procedure;
```

A.2 Double planes

A.2.1 $K^2 = 8, 6, 4, 2$ and $g = 5, 4, 3, 2$

Here we have the computations of Section 3.1, with details for the case of a double plane with $p_g = q = 1$ and $K^2 = 6$.

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![7/5,4/5],A![7/5,-4/5],A![2,1],A![2,-1],Origin(A)];
> d:=6;m1:=[2,2,2,2,2];m2:=[2,2,1,1];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m1,m2,td,~L);
> #Sections(L);BaseComponent(L);
5
Scheme over Rational Field defined by
1
```

We are using symmetry: we want to find $p_5 = (x, y)$ and $p_6 = (x, -y)$.

```
> R<x,y,b,c,d,e,n>:=PolynomialRing(Rationals(),7);
> h:=hom<PolynomialRing(L)->R|[x,y]>;
> l:=h(Sections(L));
> F:=l[1]+b*l[2]+c*l[3]+d*l[4]+e*l[5];
> G:=Evaluate(F,y,-y);
> eqF:=2*x*y*D2(F,1,2)+x^2*D2(F,1,1)+y^2*D2(F,2,2);
> eqG:=Evaluate(eqF,y,-y);
```

The condition $eqF = 0$ is used to obtain one branch of the double point $p_5 = (x, y)$ tangent to the line T_5. In fact, the tangent cone of a plane curve $\{F(X, Y) = 0\}$ at (x, y) is given by

$$\frac{\partial^2 F}{\partial X^2}(x, y)\frac{(X - x)^2}{2} + \frac{\partial^2 F}{\partial X \partial Y}(x, y)(X - x)(Y - y) + \frac{\partial^2 F}{\partial Y^2}(x, y)\frac{(Y - y)^2}{2}.$$

One can verify that this equation is divisible by $xY - yX$ only if

$$2xy\frac{\partial^2 F}{\partial X \partial Y}(x, y) + x^2\frac{\partial^2 F}{\partial X^2}(x, y) + y^2\frac{\partial^2 F}{\partial Y^2}(x, y) = 0.$$

```
> dif:=y*(y-1/2*x)*(y-4/7*x)*(y+1/2*x)*(y+4/7*x);
```

In order to obtain $p_5, p_6 \notin T_i$, $i = 1, \ldots, 4$, and $p_5 \neq p_6$, we need dif to be *different* from zero. This is achieved by imposing the condition $1 + n \cdot dif = 0$.

```
> A:=AffineSpace(R);
> Sch:=Scheme(A,[(x-2)^2+y^2-1,F,D(F,1),D(F,2),G,D(G,1),D(G,2),\
> eqF,eqG,1+n*dif]);
> Dimension(Sch);
0
> PointsOverSplittingField(Sch);
```

This command gives the points of Sch and the necessary field extensions to define them. Choosing one of the solutions we obtain points p_5, p_6 such that there exists a pencil f_A of curves of type $15(7, (4,4)_T^5, 4)$. Let B' be a general element of f_A. The branch locus $B := B' + \sum_1^5 T_i$ is of type $20(12, (5,5)_T^5, 4)$. The corresponding minimal double plane is a surface of general type with $p_g = q = 1$, $K^2 = 6$ and $g = 4$.

Verification that $B_0 := B'$ is as stated:

```
> R<r3>:=PolynomialRing(Rationals());
> K<r3>:=NumberField(r3^4 - 570063504574501/8986626*r3^2+\
> 19467699319949145508515314101/323037787455504);
> x1:=-1225449/218906496039245*r3^2 + 6763320857703/401161\
> 4254780;
> y1:=-7879209182423568/197115095314362370162761495*r3^3 +
> 37270947258282841632/117281546566527266624785*r3;
> //
> A<x,y>:=AffineSpace(K,2);
> p:=[A![7/5,4/5],A![7/5,-4/5],A![2,1],A![2,-1],A![x1,y1],\
> A![x1,-y1],Origin(A)];
> d:=15;m1:=[4,4,4,4,4,4,7];m2:=[4,4,4,4,4];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m1,m2,td,~L);
> #Sections(L);BaseComponent(L);
2
Scheme over K defined by
1
> B0:=Curve(A,Sections(L)[1]+Sections(L)[2]);
> IsReduced(B0);
true
> Multiplicity(B0,Origin(A));
7
> IsOrdinarySingularity(B0,Origin(A));
true
> [Multiplicity(B0,p[i]):i in [1..6]];
[ 4, 4, 4, 4, 4, 4 ]
> IsOrdinarySingularity(B0,p[6]);
true
> T:=[Curve(A,y-p[i][2]/p[i][1]*x):i in [1..5]];
> [IntersectionNumber(B0,T[i],p[i]):i in [1..5]];
[ 8, 8, 8, 8, 8 ]
> ResolutionGraph(B0,p[1]);
The resolution graph on the Digraph
Vertex     Neighbours

1 ([ -2, 4, 1, 0 ]) 2 ;
2 ([ -1, 8, 2, 4 ]) ;
```

One obtains the same resolution graph for $p_2, ..., p_5$.

The singularities of a general element of L are no worst than the ones above, hence we do not need to verify that B_0 has no other singularities.

The other cases, $p_g = q = 1$, $K^2 = 8, 4, 2$ and $g = 5, 3, 2$, are analogous to the previous one. One needs only to ask Magma (using the procedure $LinSys$) for curves of type

$$[10 + 2n](2n + 2, (5, 5)_T^n, 4^{6-n}), \quad n = 6, 4, 3,$$

with singularities at the previous points $p_0, \ldots, p_6$.

A.2.2 $\quad K^2 = 7, 5, 3$ and $g = 5, 4, 3$

The detailed computations of Section 3.2 are as follows.

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![1,0],A![1/5,2/5],A![2/5,1/5],A![8/5,9/5],\
> A![9/5,8/5],Origin(A)];
> d:=15;m1:=[4,4,4,4,4,7];m2:=[4,4,4,4,4];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m1,m2,td,~L);
> #Sections(L);BaseComponent(L);
8
Scheme over Rational Field defined by
1
> Bup:=[Evaluate(Sections(L)[i],y,(x-1)*y+0) div (x-1)^4:\
> i in [1..#Sections(L)]];
> L:=LinearSystem(A,Bup);
> Bup:=[Evaluate(Sections(L)[i],y,(x-1)*y+0) div (x-1)^4:\
> i in [1..#Sections(L)]];
> L1:=LinearSystem(A,Bup);
```

At this stage we have imposed the $(4, 4)$-points and resolved p_1.

```
> R<x,y,u,v,n>:=PolynomialRing(Rationals(),5);
> h:=hom<PolynomialRing(L1)->R|[x,y]>;
> l:=h(Sections(L1));
```

Now we impose, to the elements of l, the necessary conditions in order to obtain the $(3, 3)$-point $p_6 = (u, v)$. The matrix Mt defined by these conditions cannot have maximal rank.

```
> H:=[Evaluate(l[i],[u,v,u,v,n]):i in [1..#l]];
> F:=[Evaluate(l[i],y,(x-u)*y+v):i in [1..#l]];
> G:=[(F[i]-Evaluate(F[i],x,u)) div (x-u):i in [1..#l]];
> G1:=[(G[i]-Evaluate(G[i],x,u)) div (x-u):i in [1..#l]];
> G2:=[(G1[i]-Evaluate(G1[i],x,u)) div (x-u):i in [1..#l]];
> F:=G2;
> M:=[[H[i],D(H[i],3),D(H[i],4),D2(H[i],3,3),D2(H[i],3,4),\
> D2(H[i],4,4),F[i],D(F[i],1),D(F[i],2),D2(F[i],1,1),\
> D2(F[i],1,2),D2(F[i],2,2)]:i in [1..#l]];
> ME:=[[Evaluate(M[i][o],[1,y,1,v,n]):o in [1..12]]:i in [1..#M]];
```

```
> //This last step is needed to increase the speed of calculations.
> Mt:=Matrix(ME);
> min:=Minors(Mt,#l);
> A:=AffineSpace(R);
> Sch:=Scheme(A,min cat [x-1,u-1,1+n*v]);
> Dimension(Sch);
0
> PointsOverSplittingField(Sch);
```

As before, this gives various solutions. We choose one who works. Here goes the verifications:

```
> R<r2>:=PolynomialRing(Rationals());
> K<r2>:=NumberField(r2^2 - 1292/35);
> A<x,y>:=AffineSpace(K,2);
> p:=[A![1/5,2/5],A![2/5,1/5],A![8/5,9/5],A![9/5,8/5],\
> Origin(A)];
> d:=15;m1:=[4,4,4,4,7];m2:=[4,4,4,4];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m1,m2,td,~L);
> //
> q:=A![1,0];m:=[4,4,3,3];
> td:=[0,35/1292*r2,-455/15504*r2 + 455/7752];
> LinSys2(A,L,q,m,td,~J);
> #Sections(J);
1
> B0:=Curve(A,Sections(J)[1]);
> IsReduced(B0);
true
> p1:=[q] cat p;
> T:=[Curve(A,y-p1[i][2]/p1[i][1]*x):i in [1..5]];
> [Multiplicity(B0,p1[i]):i in [1..6]];
[ 4, 4, 4, 4, 4, 7 ]
> [IntersectionNumber(T[i],B0,p1[i]):i in [1..5]];
[ 8, 8, 8, 8, 8 ]
> [ResolutionGraph(B0,p1[i]):i in [1,2]];
[
    The resolution graph on the Digraph
    Vertex      Neighbours

    1 ([ -2, 8, 2, 1 ]) 2 3 ;
    2 ([ -2, 4, 1, 0 ]) ;
    3 ([ -2, 11, 3, 0 ])        4 ;
    4 ([ -1, 14, 4, 3 ])        ;
    ,
    The resolution graph on the Digraph
    Vertex      Neighbours

    1 ([ -2, 4, 1, 0 ]) 2 ;
    2 ([ -1, 8, 2, 4 ]) ;
```

```
]
```

The resolution graphs for the points p_3, p_4 and p_5 are equal to this last one.

Now we calculate the pencil J which induces the Albanese fibration.

```
> d:=16;m1:=[4,4,4,4,8];m2:=[4,4,4,4];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m1,m2,td,~L);
> q:=A![1,0];m:=[4,4,4,4];
> td:=[0,35/1292*r2,-455/15504*r2 + 455/7752];
> LinSys2(A,L,q,m,td,~J);
> #Sections(J);BaseComponent(J);
2
Scheme over K defined by
1
> Jy:=LinearSystem(J,Curve(A,y));
> B0 eq Curve(A,Sections(Jy)[1] div y);
true
```

With this we have constructed a minimal double plane with $p_g = q = 1$, $K^2 = 7$ and $g = 5$.

A.2.3 $K^2 = 6$ and $g = 3$

Here we give the detailed computations of Section 3.3.

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![4/5,7/5],A![-4/5,7/5],A![1,2],A![-1,2],Origin(A)];
> d:=6;m1:=[2,2,2,2,2];m2:=[2,2,2,2];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m1,m2,td,~L);
> #Sections(L);BaseComponent(L);
2
Scheme over Rational Field defined by
1
> R<x,y,b,n>:=PolynomialRing(Rationals(),4);
> h:=hom<PolynomialRing(L)->R|[x,y]>;
> l:=h(Sections(L));
> F:=l[1]+b*l[2];
> G:=Evaluate(F,x,-x);
> C:=x^2+(y-2)^2-1;
> //
> eqF:=D(C,1)*D(F,2)-D(C,2)*D(F,1);//To obtain a curve
> //tangent to the conic C at p_5=(x,y).
> eqG:=Evaluate(eqF,x,-x);//The same to p_6=(-x,y).
> dif:=x*(y-2*x)*(y-7/4*x)*(y+2*x)*(y+7/4*x);
> //We need dif to be non-zero.
> //
> A:=AffineSpace(R);
> Sch:=Scheme(A,[C,F,G,eqF,eqG,1+n*D(F,1)*D(F,2)*dif]);
> Dimension(Sch);
```

```
0
> PointsOverSplittingField(Sch);
{@(-r2, 0, 5377/5292, -3/307328), (r2, 0, 5377/5292, -3/307328)@}
Algebraically closed field with 2 variables
Defining relations: [
    r2^2 + 3,
    r1^2 + 3 ]
```

This gives the points p_5, p_6. We omit the remaining verifications (they are similar to the ones in A.2.1 and A.2.2). We notice only that, to verify that a curve D has no singularity at infinity, one can proceed as follows:

```
> PD:=ProjectiveClosure(D);
> Dimension(SingularSubscheme(PD) meet LineAtInfinity(A));
```

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